Magnetic Branes in Gauss-Bonnet Gravity

M. H. Dehghani*

Physics Department and Biruni Observatory,

Shiraz University, Shiraz 71454, Iran

and

Institute for Studies in Theoretical Physics and Mathematics (IPM)

P.O. Box 19395-5531, Tehran, Iran

We present two new classes of magnetic brane solutions in Einstein-Maxwell-Gauss-Bonnet gravity with a negative cosmological constant. The first class of solutions yields an $(n+1)$-dimensional spacetime with a longitudinal magnetic field generated by a static magnetic brane. We also generalize this solution to the case of spinning magnetic branes with one or more rotation parameters. We find that these solutions have no curvature singularity and no horizons, but have a conic geometry. In these spacetimes, when all the rotation parameters are zero, the electric field vanishes, and therefore the brane has no net electric charge. For the spinning brane, when one or more rotation parameters are non zero, the brane has a net electric charge which is proportional to the magnitude of the rotation parameter. The second class of solutions yields a spacetime with an angular magnetic field. These solutions have no curvature singularity, no horizon, and no conical singularity. Again we find that the net electric charge of the branes in these spacetimes is proportional to the magnitude of the velocity of the brane. Finally, we use the counterterm method in the Gauss-Bonnet gravity and compute the conserved quantities of these spacetimes.

I. INTRODUCTION

The possibility that spacetime may have more than four dimensions is now a standard assumption in high energy physics. From a cosmological point of view, our observable

*Electronic address: dehghani@physics.susc.ac.ir

Universe may be viewed as a brane embedded into a higher dimensional spacetime. In the context of string theory, extra dimensions were promoted from an interesting curiosity to a theoretical necessity since superstring theory requires a ten-dimensional spacetime to be consistent from the quantum point of view. The idea of brane cosmology is also consistent with string theory, which suggests that matter and gauge interaction (described by an open string) may be localized on a brane, embedded into a higher dimensional spacetime. The field represented by closed strings, in particular, gravity, propagate in the whole of spacetime.

This underscores the need to consider gravity in higher dimensions. In this context one may use another consistent theory of gravity in any dimension with a more general action. This action may be written, for example, through the use of string theory. The effect of string theory on classical gravitational physics is usually investigated by means of a low energy effective action which describes gravity at the classical level [1]. This effective action consists of the Einstein-Hilbert action plus curvature-squared terms and higher powers as well, and in general give rise to fourth order field equations and bring in ghosts. However, if the effective action contains the higher powers of curvature in particular combinations, then only second order field equations are produced and consequently no ghosts arise [2]. The effective action obtained by this argument is precisely of the form proposed by Lovelock [3]. The appearance of higher derivative gravitational terms can be seen also in the renormalization of quantum field theory in curved spacetime [4].

In this paper we want to restrict ourself to the first three terms of Lovelock gravity. The first two terms are the Einstein-Hilbert term with cosmological constant, while the third term is known as the Gauss-Bonnet term. This term appears naturally in the next-to-leading order term of the heterotic string effective action and plays a fundamental role in Chern-Simons gravitational theories [5]. From a geometric point of view, the combination of the Einstein-Gauss-Bonnet terms constitutes, for five-dimensional spacetimes, the most general Lagrangian producing second order field equations, as in the four-dimensional gravity where the Einstein-Hilbert action is the most general Lagrangian producing second order field equations [6].

These facts provide a strong motivation for considering new exact solutions of the Einstein-Gauss-Bonnet gravity. Because of the nonlinearity of the field equations, it is very difficult to find out nontrivial exact analytical solutions of Einstein's equation with higher curvature terms. In most cases, one has to adopt some approximation methods or

find solutions numerically. However, static spherically symmetric black hole solutions of the Gauss-Bonnet gravity were found in Ref. [7]. Black hole solutions with nontrivial topology in this theory were also studied in Refs. [8, 9, 10]. The thermodynamics of charged static spherically symmetric black hole solutions was considered in [11]. All of these known solutions are static. Recently I introduced a new class of asymptotically anti-de Sitter rotating black brane solutions in the Einstein-Gauss-Bonnet gravity and considered its thermodynamics [12].

In this paper we are dealing with the issue of the spacetimes generated by static, spinning, and traveling brane sources in $(n+1)$-dimensional Einstein-Maxwell-Gauss-Bonnet gravity that are horizonless and have nontrivial external solutions. These kinds of solutions have been investigated by many authors in the context of Einstein gravity. Static uncharged cylindrically symmetric solutions of Einstein gravity in four dimensions were considered in [13]. Similar static solutions in the context of cosmic string theory were found in [14]. All of these solutions [13, 14] are horizonless and have a conical geometry, which are everywhere flat except at the location of the line source. An extension to include the electromagnetic field has also been done [15, 16]. The generalization of the four-dimensional solution found in [16] to the case of $(n+1)$-dimensional solution with all rotation and boost parameters has been done in [17]. Some solutions of type IIB supergravity compactified on a four-dimensional torus have been considered in [18], which have no curvature singularity and no conic singularity. Here we will find these kinds of solutions in the Gauss-Bonnet gravity, and use the counterterm method to compute the conserved quantities of the system.

The outline of our paper is as follows. We give a brief review of the field equations in Sec. II. In Sec. III we first present a new class of static horizonless solutions which produce longitudinal magnetic field, and then endow these spacetime solutions with a global rotation. We also generalize these rotating solutions to the case of spacetimes with more rotation parameters. In Sec. IV we introduce those horizonless solutions that produce an angular magnetic field. Section V will be devoted to the use of the counterterm method to compute the conserved quantities of these spacetimes. We also show that the electric charge densities of the branes with rotation or a boost parameter are proportional to the magnitude of the boost parameters. We finish our paper with some concluding remarks.

II. FIELD EQUATIONS IN EINSTEIN-MAXWELL-GAUSS-BONNET GRAVITY

The most fundamental assumption in standard general relativity are the requirement of general covariance and that the field equations be second order. Based on these principles, the most general Lagrangian in arbitrary dimensions is the Lovelock Lagrangian. The Lagrangian of the Lovelock theory, which is the sum of dimensionally extended Euler densities, may be written as

$$\mathcal{L}_G = \frac{1}{2}\sum_i^n c_i \mathcal{L}_i, \tag{1}$$

where c_i is an arbitrary constant and $\mathcal{L}_i$ is the Euler density of a $2i$-dimensional manifold,

$$\mathcal{L}_i = (-2)^{-i}\delta^{a_1 b_1 \ldots a_i b_i}_{c_1 d_1 \ldots c_i d_i} R^{c_1 d_1}_{a_1 b_1} \ldots . R^{c_i d_i}_{a_i b_i}. \tag{2}$$

Here the first two terms $c_0\mathcal{L}_0 = 2\Lambda$, where Λ is the cosmological constant, and $c_1\mathcal{L}_1 = R$ give us the Einstein-Hilbert term and $c_2\mathcal{L}_2 = \alpha(R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2)$ is the Gauss-Bonnet term. Thus, the gravitational action of an $(n+1)$-dimensional asymptotically anti-de Sitter spacetimes with the Gauss-Bonnet term in the presence of an electromagnetic field, in a unit system in which $8\pi G = 1$, can be written as

$$I_g = \frac{1}{2}\int d^{n+1}x\sqrt{-g}\{R + \frac{n(n-1)}{l^2} + \alpha(R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2) + F_{\mu\nu}F^{\mu\nu}\}, \tag{3}$$

where R, $R_{\mu\nu\rho\sigma}$, and $R_{\mu\nu}$ are the Ricci scalar and Riemann and Ricci tensors of the spacetime, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor field, and A_μ is the vector potential. α is the Gauss-Bonnet coefficient with dimension $(length)^2$ and is positive in the heterotic string theory [7]. So we restrict ourselves to the case $\alpha \geq 0$. Varying the action over the metric tensor $g_{\mu\nu}$ and electromagnetic field $F_{\mu\nu}$, the equations of gravitational and electromagnetic fields are obtained as

$$\begin{aligned} &R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{n(n-1)}{2l^2}g_{\mu\nu} - \alpha\{\frac{1}{2}g_{\mu\nu}(R_{\gamma\delta\lambda\sigma}R^{\gamma\delta\lambda\sigma} - 4R_{\gamma\delta}R^{\gamma\delta} + R^2) \\ &-2RR_{\mu\nu} + 4R_{\mu\gamma}R^{\gamma}{}_{\nu} + 4R_{\gamma\delta}R^{\gamma\ \delta}_{\mu\nu} - 2R_{\mu\gamma\delta\lambda}R_{\nu}{}^{\gamma\delta\lambda}\} = T_{\mu\nu}, \end{aligned} \tag{4}$$

$$\nabla_\mu F_{\mu\nu} = 0, \tag{5}$$

where $T_{\mu\nu}$ is the electromagnetic stress tensor

$$T_{\mu\nu} = 2F^{\lambda}{}_{\mu}F_{\lambda\nu} - \frac{1}{2}F_{\lambda\sigma}F^{\lambda\sigma}g_{\mu\nu}. \tag{6}$$

Equation (4) does not contain the derivative of the curvatures, and therefore the derivatives of the metric higher than two do not appear. Thus, the Gauss-Bonnet gravity is a special case of higher derivative gravity.

III. THE LONGITUDINAL MAGNETIC FIELD SOLUTIONS

Here we want to obtain the $(n+1)$-dimensional solutions of Eqs. (4)-(6) which produce longitudinal magnetic fields in the Euclidean submanifold spans by the x^i coordinates ($i = 1, ..., n-2$). We assume that the metric has the following form:

$$ds^2 = -\frac{\rho^2}{l^2}dt^2 + \frac{d\rho^2}{f(\rho)} + l^2 f(\rho) d\phi^2 + \frac{\rho^2}{l^2}\sum_{i=1}^{n-2}(dx^i)^2. \tag{7}$$

Note that the coordinates x^i have the dimension of length, while the angular coordinate ϕ is dimensionless as usual and ranges in $0 \leq \phi < 2\pi$. The motivation for this metric gauge [$g_{tt} \propto -\rho^2$ and $(g_{\rho\rho})^{-1} \propto g_{\phi\phi}$] instead of the usual Schwarzschild gauge [$(g_{\rho\rho})^{-1} \propto g_{tt}$ and $g_{\phi\phi} \propto \rho^2$] comes from the fact that we are looking for a magnetic solution instead of an electric one. In this section we want to consider only the magnetically charged case which produces a longitudinal magnetic field in the Euclidean submanifold spanned by the x^i coordinates. In the next section we consider the angular magnetic solutions. Thus, one may assume that

$$A_\mu = h(\rho)\delta_\mu^\phi. \tag{8}$$

For purely electrically charged solutions in Gauss-Bonnet gravity, see [12]. The functions $f(\rho)$ and $h(\rho)$ may be obtained by solving the field equations (4) and (5). Using Eq. (5) one obtains

$$\rho\frac{\partial^2 h}{\partial \rho^2} + (n-1)\frac{\partial h}{\partial \rho} = 0. \tag{9}$$

Thus, $h(\rho) = C_1/\rho^{(n-2)}$, where C_1 is an arbitrary real constant. To get the solution of the Einstein-Maxwell equation in the case of $\alpha = 0$, which I introduced in [17], one should choose the arbitrary constant $C_1 = -2ql^{n-1}/(n-2)$. To find the function $f(\rho)$, one may use any components of Eq. (4). The simplest equation is the $\rho\rho$ component of these equations which can be written as

$$\begin{aligned}&(n-1)\{l^2\rho^{2n-5}[2(n-2)(n-3)\alpha f - \rho^2]f' + n\rho^{2n-2} \\ &+(n-2)l^2\rho^{2n-6}[(n-3)(n-4)\alpha f - \rho^2]f\} + 8q^2l^{2n-2} = 0,\end{aligned} \tag{10}$$

where the prime denotes a derivative with respect to the ρ coordinate. The solutions of Eq. (10) can be written as

$$f(\rho)=\frac{\rho^2}{2(n-2)(n-3)\alpha}\left\{1\pm\sqrt{1-\frac{4(n-2)(n-3)\alpha}{l^2}\left(1+\frac{C_2}{\rho^n}-\frac{8q^2l^{2n-2}}{(n-1)(n-2)\rho^{2(n-1)}}\right)}\right\}, \tag{11}$$

where C_2 is an arbitrary constant. As one can see from Eq. (11), the solution has two branches with "$-$" and "$+$" signs. The arbitrary constant C_2 should be chosen such that the solution obtained reduces to that of the Einstein-Maxwell equation introduced in [17] as α goes to zero. In order to have the desired function, we should choose the branch with the "$-$" sign and fix $C_2=-8ml^n$. The parameters m and q are the mass and charge parameters and r_+ is the largest positive real solution of $f(\rho)=0$.

In order to study the general structure of this solution, we first look for curvature singularities. It is easy to show that the Kretschmann scalar $R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}$ diverges at $\rho=0$ and therefore one might think that there is a curvature singularity located at $\rho=0$. However, as will be seen below, the spacetime will never achieve $\rho=0$. Now we look for the existence of horizons, and therefore we look for possible black brane solutions. One should conclude that there are no horizons and therefore no black brane solutions. The horizons, if any exist, are given by the zeros of the function $f(\rho)=g_{\rho\rho}^{-1}$. Let us denote the zeros of $f(\rho)$ by r_+. The function $f(\rho)$ is negative for $\rho<r_+$ and positive for $\rho>r_+$, and therefore one may think that the hypersurface of constant time and $\rho=r_+$ is the horizon. However, this analysis is not correct. Indeed, one may note that $g_{\rho\rho}$ and $g_{\phi\phi}$ are related by $f(\rho)=g_{\rho\rho}^{-1}=l^{-2}g_{\phi\phi}$, and therefore when $g_{\rho\rho}$ becomes negative (which occurs for $\rho<r_+$) so does $g_{\phi\phi}$. This leads to an apparent change of signature of the metric from $(n-1)+$ to $(n-2)+$, and therefore indicates that we are using an incorrect extension. To get rid of this incorrect extension, we introduce the new radial coordinate r as

$$r^2=\rho^2-r_+^2\Rightarrow d\rho^2=\frac{r^2}{r^2+r_+^2}dr^2. \tag{12}$$

With this new coordinate, the metric (7) is

$$\begin{aligned}ds^2 &= -\frac{r^2+r_+^2}{l^2}dt^2+l^2f(r)d\phi^2\\ &\quad +\frac{r^2}{(r^2+r_+^2)f(r)}dr^2+\frac{r^2+r_+^2}{l^2}dX^2,\end{aligned} \tag{13}$$

where dX^2 is the Euclidean metric on the $(n-2)$-dimensional submanifold, the coordinates r and ϕ assume the values $0 \leq r < \infty$ and $0 \leq \phi < 2\pi$, and $f(r)$ is now given as

$$f(r) = \frac{r^2 + r_+^2}{2(n-2)(n-3)\alpha}\{1 - \sqrt{1 - \frac{4(n-2)(n-3)\alpha}{l^2}\left(1 - \frac{8ml^n}{(r^2+r_+^2)^{n/2}} - \frac{8q^2l^{2n-2}}{(n-1)(n-2)(r^2+r_+^2)^{n-1}}\right)}\}. \quad (14)$$

The gauge potential in the new coordinate is

$$A_\mu = -\frac{2}{(n-2)}\frac{ql^{n-1}}{(r^2+r_+^2)^{(n-2)/2}}\delta_\mu^\phi. \quad (15)$$

The function $f(r)$ given in Eq. (14) is positive in the whole spacetime and is zero at $r = 0$. Also note that the Kretschmann scalar does not diverge in the range $0 \leq r < \infty$. Therefore this spacetime has no curvature singularities and no horizons. However, it has a conic geometry and has a conical singularity at $r = 0$. In fact, using a Taylor expansion, in the vicinity of $r = 0$ the metric (13) is

$$\begin{aligned} ds^2 &= -\frac{r_+^2}{l^2}dt^2 + 2\frac{l^2}{r_+^2}\left(n + \frac{4q^2l^{2n-2}}{(n-1)r_+^{2n-2}}\right)^{-1}dr^2 \\ &+\frac{1}{2l^2}\left(n + \frac{4q^2l^{2n-4}}{(n-1)r_+^{2n-2}}\right)r^2d\phi^2 + \frac{r_+^2}{l^2}dX^2, \end{aligned} \quad (16)$$

which clearly shows that the spacetime has a conical singularity at $r = 0$. It is worthwhile to mention that the magnetic solutions obtained here have distinct properties relative to the electric solutions obtained in [12]. Indeed, the electric solutions have black holes, while the magnetic do not.

Of course, one may ask for the completeness of the spacetime with $r \geq 0$ [16, 19]. It is easy to see that the spacetime described by Eq. (13) is both null and timelike geodesically complete for $r \geq 0$. To do this, one may show that every null or timelike geodesic starting from an arbitrary point either can be extended to infinite values of the affine parameter along the geodesic or will end on a singularity at $r = 0$. Using the geodesic equation, one obtains

$$\dot{t} = \frac{l^2}{r^2 + r_+^2}E, \quad \dot{x^i} = \frac{l^2}{r^2 + r_+^2}P^i, \quad \dot{\phi} = \frac{1}{l^2 f(r)}L, \quad (17)$$

$$r^2\dot{r}^2 = (r^2 + r_+^2)f(r)\left[\frac{l^2(E^2 - \mathbf{P}^2)}{r^2 + r_+^2} - \alpha\right] - \frac{r^2 + r_+^2}{l^2}L^2, \quad (18)$$

where the overdot denotes the derivative with respect to an affine parameter, and α is zero for null geodesics and $+1$ for timelike geodesics. E, L, and P^i's are the conserved quantities associated with the coordinates t, ϕ, and x^i's, respectively, and $\mathbf{P}^2 = \sum_{i=1}^{n-2}(P^i)^2$. Notice that $f(r)$ is always positive for $r > 0$ and zero for $r = 0$.

First we consider the null geodesics ($\alpha = 0$). (i) If $E^2 > \mathbf{P}^2$ the spiraling particles ($L > 0$) coming from infinity have a turning point at $r_{tp} > 0$, while the nonspiraling particles ($L = 0$) have a turning point at $r_{tp} = 0$. (ii) If $E = \mathbf{P}$ and $L = 0$, whatever the value of r, $\dot{r}$ and $\dot{\phi}$ vanish and therefore the null particles moves in a straight line in the $(n-2)$-dimensional submanifold spanned by x^1 to x^{n-2}. (iii) For $E = \mathbf{P}$ and $L \neq 0$, and also for $E^2 < \mathbf{P}^2$ and any values of L, there is no possible null geodesic.

Now, we analyze the timelike geodesics ($\alpha = +1$). Timelike geodesics is possible only if $l^2(E^2 - \mathbf{P}^2) > r_+^2$. In this case spiraling ($L \neq 0$) timelike particles are bound between r_{tp}^a and r_{tp}^b given by

$$0 < r_{tp}^a \leq r_{tp}^b < \sqrt{l^2(E^2 - \mathbf{P}^2) - r_+^2}, \tag{19}$$

while the turning points for the nonspiraling particles ($L = 0$) are $r_{tp}^1 = 0$ and $r_{tp}^2 = \sqrt{l^2(E^2 - \mathbf{P}^2) - r_+^2}$. Thus, we confirmed that the spacetime described by Eq. (13) is both null and timelike geodesically complete.

When m and q are zero, the vacuum solution is

$$f(\rho) = \frac{\rho^2}{2(n-2)(n-3)\alpha}\left(1 - \sqrt{1 - \alpha\frac{4(n-2)(n-3)}{l^2}}\right). \tag{20}$$

Equation (20) shows that for a positive value of α, this parameter should be less than $\alpha \leq l^2/4(n-2)(n-3)$. Also note that the AdS solution of the theory has the effective cosmological constant

$$l_{\text{eff}}^2 = 2(n-2)(n-3)\alpha\left(1 - \sqrt{1 - \frac{4(n-2)(n-3)\alpha}{l^2}}\right)^{-1}. \tag{21}$$

We use the effective cosmological constant (21) later in order to introduce the counterterm for the action.

A. The rotating longitudinal solutions

Now, we want to endow our spacetime solution (13) with a global rotation. In order to add angular momentum to the spacetime, we perform the following rotation boost in the

$t-\phi$ plane

$$t \mapsto \Xi t - a\phi, \qquad \phi \mapsto \Xi\phi - \frac{a}{l^2}t, \tag{22}$$

where a is a rotation parameter and $\Xi = \sqrt{1+a^2/l^2}$. Substituting Eq. (22) into Eq. (13) we obtain

$$\begin{aligned} ds^2 &= -\frac{r^2+r_+^2}{l^2}\left(\Xi dt - ad\phi\right)^2 + \frac{r^2dr^2}{(r^2+r_+^2)f(r)} \\ &+l^2 f(r)\left(\frac{a}{l^2}dt - \Xi d\phi\right)^2 + \frac{r^2+r_+^2}{l^2}dX^2, \end{aligned} \tag{23}$$

where $f(r)$ is the same as $f(r)$ given in Eq. (14). The gauge potential is now given by

$$A_\mu = \frac{2}{(n-2)}\frac{ql^{(n-3)}}{(r^2+r_+^2)^{(n-2)/2}}\left(a\delta_\mu^0 - l^2\Xi\delta_\mu^\phi\right). \tag{24}$$

The transformation (22) generates a new metric, because it is not a permitted global coordinate transformation [20]. This transformation can be done locally but not globally. Therefore, the metrics (13) and (23) can be locally mapped into each other but not globally, and so they are distinct. Note that this spacetime has no horizon and curvature singularity. However, it has a conical singularity at $r=0$. One should note that these solutions are different from those discussed in [12], which were electrically charged rotating black brane solutions in Gauss-Bonnet gravity. The electric solutions have black holes, while the magnetic do not. It is worthwhile to mention that this solution reduces to the solution of Einstein-Maxwell equation introduced in [17] as α goes to zero.

B. The general rotating longitudinal solution with more rotation parameters

For the sake of completeness we give the general rotating longitudinal solution with more rotation parameters. The rotation group in $n+1$ dimensions is $SO(n)$ and therefore the number of independent rotation parameters is $[(n+1)/2]$, where $[x]$ is the integer part of x. We now generalize the above solution given in Eq. (23) with $k \leq [(n+1)/2]$ rotation parameters. This generalized solution can be written as

$$\begin{aligned} ds^2 &= -\frac{r^2+r_+^2}{l^2}\left(\Xi dt - \sum_{i=1}^{k} a_i d\phi^i\right)^2 + f(r)\left(\sqrt{\Xi^2-1}dt - \frac{\Xi}{\sqrt{\Xi^2-1}}\sum_{i=1}^{k} a_i d\phi^i\right)^2 \\ &+\frac{r^2dr^2}{(r^2+r_+^2)f(r)} + \frac{r^2+r_+^2}{l^2(\Xi^2-1)}\sum_{i<j}^{k}(a_i d\phi_j - a_j d\phi_i)^2 + \frac{r^2+r_+^2}{l^2}dX^2, \end{aligned} \tag{25}$$

where $\Xi = \sqrt{1+\sum_i^k a_i^2/l^2}$, dX^2 is the Euclidean metric on the $(n-k-1)$-dimensional submanifold and $f(r)$ is the same as $f(r)$ given in Eq. (14). The gauge potential is

$$A_\mu = \frac{2}{(n-2)} \frac{q l^{(n-2)}}{(r^2+r_+^2)^{(n-2)/2}} \left(\sqrt{\Xi^2-1}\delta_\mu^0 - \frac{\Xi}{\sqrt{\Xi^2-1}} a_i \delta_\mu^i \right); \quad \text{(no sum on } i\text{)}. \tag{26}$$

Again this spacetime has no horizon and curvature singularity. However, it has a conical singularity at $r=0$. One should note that these solutions reduce to those discussed in [12].

IV. THE ANGULAR MAGNETIC FIELD SOLUTIONS

In Sec. III we found a spacetime generated by a magnetic source which produces a longitudinal magnetic field along x^i coordinates. In this section we want to obtain a spacetime generated by a magnetic source that produce angular magnetic fields along the ϕ^i coordinates. Following the steps of Sec. III but now with the roles of ϕ and x interchanged, we can directly write the metric and vector potential satisfying the field equations (4)-(6) as

$$\begin{aligned} ds^2 &= -\frac{r^2+r_+^2}{l^2}dt^2 + \frac{r^2 dr^2}{(r^2+r_+^2)f(r)} \\ &\quad + (r^2+r_+^2)\sum_{i=1}^{n-2}(d\phi^i)^2 + f(r)dx^2, \end{aligned} \tag{27}$$

where $f(r)$ is given in Eq. (14). The angular coordinates ϕ^i's range in $0 \le \phi^i < 2\pi$. The gauge potential is now given by

$$A_\mu = -\frac{2}{(n-2)} \frac{q l^{(n-2)}}{(r^2+r_+^2)^{(n-2)/2}} \delta_\mu^x. \tag{28}$$

The Kretschmann scalar does not diverge for any r and therefore there is no curvature singularity. The spacetime (27) is also free of conic singularity. In addition, it is notable to mention that the radial geodesic passes through $r=0$ (which is free of singularity) from positive values to negative values of the coordinate r. This shows that the radial coordinate in Eq. (27) can take the values $-\infty < r < \infty$. This analysis may suggest that one is in the presence of a traversable wormhole with a throat of dimension r_+. However, in the vicinity of $r=0$, the metric (27) can be written as

$$ds^2 = -\frac{r_+^2}{l^2}dt^2 + 2\frac{l^2}{r_+^2}\left(n + \frac{4q^2 l^{2n-2}}{(n-1)r_+^{2n-2}}\right)^{-1} dr^2$$

$$+r_+^2 d\Omega^2 + \frac{1}{2l^2}\left(n + \frac{4q^2 l^{2n-4}}{(n-1)r_+^{2n-2}}\right) r^2 dx^2, \tag{29}$$

which clearly shows that, at $r = 0$, the x direction collapses and therefore we have to abandon the wormhole interpretation.

To add linear momentum to the spacetime, we perform the boost transformation [$t \mapsto \Xi t - (v/l)x$, $x \mapsto \Xi x - (v/l)t$] in the $t - x$ plane and obtain

$$\begin{aligned} ds^2 \;&=\; -\frac{r^2 + r_+^2}{l^2}\left(\Xi dt - \frac{v}{l}dx\right)^2 + f(r)\left(\frac{v}{l}dt - \Xi dx\right)^2 \\ &\quad + \frac{r^2 dr^2}{(r^2 + r_+^2) f(r)} + (r^2 + r_+^2) d\Omega^2, \end{aligned} \tag{30}$$

where v is a boost parameter and $\Xi = \sqrt{1 + v^2/l^2}$. The gauge potential is given by

$$A_\mu = \frac{2}{(n-2)} \frac{\lambda l^{(n-3)}}{(r^2 + r_+^2)^{(n-2)/2}} \left(v\delta_\mu^0 - l\Xi\delta_\mu^x\right). \tag{31}$$

Contrary to transformation (22), this boost transformation is permitted globally since x is not an angular coordinate. Thus the boosted solution (30) is not a new solution. However, it generates an electric field.

V. THE CONSERVED QUANTITIES OF A MAGNETIC ROTATING BRANE

It is well known that the gravitational action given in Eq. (3) diverges. A systematic method of dealing with this divergence in Einstein gravity is through the use of the counterterms method inspired by the anti-de Sitter conformal field theory (AdS/CFT) correspondence [21]. This conjecture, which relates the low energy limit of string theory in asymptotically anti de-Sitter spacetime and the quantum field theory living on the boundary of it, have attracted a great deal of attention in recent years. This equivalence between the two formulations means that, at least in principle, one can obtain complete information on one side of the duality by performing computation on the other side. A dictionary translating between different quantities in the bulk gravity theory and their counterparts on the boundary has emerged, including the partition functions of both theories. This conjecture is now a fundamental concept that furnishes a means for calculating the action and conserved quantities intrinsically without reliance on any reference spacetime [22, 23, 24]. It has also been applied to the case of black holes with constant negative or zero curvature horizons

[25] and rotating higher genus black branes [26]. Although the AdS/CFT correspondence applies for the case of a specially infinite boundary, it was also employed for the computation of the conserved and thermodynamic quantities in the case of a finite boundary [27]. The counterterm method has also been extended to the case of asymptotically de Sitter spacetimes [28].

All of the work mention in the last paragraph was limited to Einstein gravity where the universal and widely accepted Gibbons-Hawking boundary term [29] is known. In Einstein gravity the Gibbons-Hawking term, which is the trace of the extrinsic curvature of the boundary, will be added to the action in order to have a well-defined variational principle. The main difference of higher derivative gravity from Einstein gravity is that the boundary term which make the variational principle well-behaved is much more complicated [30, 31]. However, the boundary term that makes the variational principle well behaved is known for the case of Gauss-Bonnet gravity [32, 33]. Here we want to apply the counterterm method to the case of magnetic brane solutions introduced in this paper. In order to do this we choose the following counterterm

$$I_b = I_b^{(1)} + I_b^{(2)}, \tag{32}$$

where $I_b^{(1)}$ is given as [32, 33]

$$I_b^{(1)} = \int_{\delta\mathcal{M}} d^n x \sqrt{-\gamma} \left\{ K + 2\alpha \left[\frac{1}{n} \left(+3KK_{cd}K^{cd} - 2K_{ac}K^{cd}K_d{}^a - K^3 \right) - 2\widehat{G}^{ab}K_{ab} \right] \right\}. \tag{33}$$

In Eq. (33), K is the trace of the extrinsic curvature K_{ab} of any boundary $\partial\mathcal{M}$ of the manifold $\mathcal{M}$, with induced metric γ_{ab}, and $\widehat{G}^{ab}$ is the n-dimensional Einstein tensor corresponding to the metric γ_{ab}. The second boundary term $I_b^{(2)}$ is the counterterm that cancels the divergence appearing in the limit of $r \to \infty$. This term is given only in terms of the boundary quantities which do not affect the variational principle. Note that for our spacetimes the Riemann curvature of the boundary is zero and therefore $I_b^{(2)}$ has only one term given as

$$I_b^{(2)} = -\int_{\delta\mathcal{M}} d^n x \sqrt{-\gamma} \left(\frac{n-1}{l_{\text{eff}}} \right), \tag{34}$$

where l_{eff} is given in Eq. (21). One may note that this counterterm has exactly the same form as the counterterm in Einstein gravity for a spacetime with zero curvature boundary in which l is replaced by l_{eff}.The total action can be written as a linear combination of the gravity term (3), and the counterterms (33) and (34)

$$I = I_g + I_b. \tag{35}$$

It is worthwhile to mention that the action (35) has no r divergence in various dimensions for spacetimes with zero curvature boundary. However, the boundary term introduced in [30], does not have this property. In fact, the coefficients of various terms in the action of [30] in different dimension should be chosen different, in order to remove the r-divergence of the action. Having the total finite action, one can use the Brown-York definition of stress-energy tensor [34] to construct a divergence-free stress-energy tensor. Note that the last term in Eq. (33) is zero, for our case, and therefore one may write

$$\begin{aligned} T^{ab} &= \sqrt{-\gamma}\{(K\gamma^{ab} - K^{ab}) \qquad (36) \\ &+\frac{2\alpha}{n}\left(3K^2K^{ab} - 3K_{cd}K^{cd}K^{ab} + a_n KK^{ac}K_c{}^b + b_n K^{ac}K_{cd}K^{db}\right) \\ &+\frac{2\alpha}{n}\left(3KK_{cd}K^{cd} - 2K_{ac}K^{cd}K_{db} - K^3\right)\gamma^{ab} - \left(\frac{n-1}{l_{\text{eff}}}\right)\gamma^{ab}\}. \qquad (37) \end{aligned}$$

One may note that when α goes to zero, the stress-energy tensor (**??**) reduces to that of Einstein gravity.

The conserved quantity associated with a Killing vector ξ^a is

$$\mathcal{Q}(\xi) = \int_{\mathcal{B}} d^n x\sqrt{\sigma}u^a T_{ab}\xi^b, \qquad (38)$$

where $\mathcal{B}$ is the hypersurface of fixed r and t, u^a is the unit normal vector on $\mathcal{B}$, and σ is the determinant of the metric σ_{ij}, appearing in the ADM-like decomposition of the boundary metric,

$$ds^2 = -N^2dt^2 + \sigma_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \qquad (39)$$

In Eq. (39), N and N^i are the lapse and shift functions, respectively. For the spacetimes introduced in this paper, the $(n-1)$-dimensional boundaries $\mathcal{B}$ have timelike Killing vector ($\xi = \partial/\partial t$), rotational Killing vector ($\zeta_i = \partial/\partial\phi^i$), and translational Killing vector ($\varsigma_i = \partial/\partial x^i$). Thus, one obtains the conserved mass, angular momentum, and linear momentum of the system enclosed by the boundary $\mathcal{B}$ as

$$M = \int_{\mathcal{B}} d^{n-1}x\sqrt{\sigma}T_{ab}u^a\xi^b, \qquad (40)$$

$$J_i = \int_{\mathcal{B}} d^{n-1}x\sqrt{\sigma}T_{ab}u^a\zeta_i^b, \qquad (41)$$

$$P_i = \int_{\mathcal{B}} d^{n-1}x\sqrt{\sigma}T_{ab}u^a\varsigma_i^b. \qquad (42)$$

We now apply this counterterm method to the case of five-dimensional spacetimes (23) and (30). It is easy to verify that the r divergence of the action is removed by the counterterm

(34). The divergence terms of the mass of the spacetimes (23) and (30) in five dimensions will be removed if one choose $a_4 = -4$ and $b_4 = -8$. Using these coefficients together with Eqs. (40) and (41), the mass and the angular momentum densities of the spacetime (23) in five dimensions can be calculated as

$$M = m\left[4(\Xi^2 - 1) + 1\right], \tag{43}$$

$$J = -16\Xi ma. \tag{44}$$

The mass of the spacetime (30) in five dimensions is the same as (43), its angular momentum is zero, and its linear momentum is

$$P = -16\Xi mv. \tag{45}$$

Of course, one can apply this method to compute the mass, and angular and linear momenta of the spacetime in various dimensions.

Next, we calculate the electric charge of the solutions. To determine the electric field we should consider the projections of the electromagnetic field tensor on special hypersurfaces. The normal to such hypersurfaces for spacetimes with longitudinal magnetic field is

$$u^0 = \frac{1}{N}, \quad u^r = 0, \quad u^i = -\frac{N^i}{N}, \tag{46}$$

and the electric field is $E^\mu = g^{\mu\rho}F_{\rho\nu}u^\nu$. Then the electric charge density Q of the spacetimes (23) and (30) can be found by calculating the flux of the electromagnetic field at infinity, yielding

$$Q = \frac{1}{4\pi}\sqrt{\Xi^2 - 1}q. \tag{47}$$

Note that the electric charge density is proportional to the rotation parameter or boost parameter and is zero for the case of static solutions.

VI. CLOSING REMARKS

In this paper, we added the Gauss-Bonnet term to the Einstein-Maxwell action with a negative cosmological constant. We introduced two classes of solutions which are asymptotically anti-de Sitter. The first class of solutions yields an $(n+1)$-dimensional spacetime with a longitudinal magnetic field [the only nonzero component of the vector potential is $A_\phi(r)$] generated by a static magnetic brane. We also found the rotating spacetime with

a longitudinal magnetic field by a rotational boost transformation. We found that these solutions have no curvature singularity and no horizons, but have conic singularity at $r = 0$. In these spacetimes, when all the rotation parameters are zero (static case), the electric field vanishes, and therefore the brane has no net electric charge. For the spinning brane, when one or more rotation parameters are nonzero, the brane has a net electric charge density which is proportional to the magnitude of the rotation parameter given by $\sqrt{\Xi^2 - 1}$. The second class of solutions yields a spacetime with angular magnetic field. These solutions have no curvature singularity, no horizon, and no conic singularity. Again, we found that the branes in these spacetimes have no net electric charge when all the boost parameters are zero. We also showed that, for the case of traveling branes with nonzero boost parameter, the net electric charge density of the brane is proportional to the magnitude of the velocity of the brane (v).

The counterterm method inspired by the AdS/CFT correspondence conjecture has been widely applied to the case of Einstein gravity. Here we applied this method to the case of Gauss-Bonnet gravity and calculated the conserved quantities of the two classes of solutions. We found that the counterterm (34) has only one term, since the boundaries of our spacetimes are curvature-free. Other related problems such as the application of the counterterm method to the case of solutions of higher curvature gravity with nonzero curvature boundary remain to be carried out.

---

[1] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, Nucl. Phys. **B258**, 46 (1985); M. B. Greens, J. H. Schwarz and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, England, 1987); D. Lust and S. Theusen, *Lectures on String Theory* (Springer, Berlin, 1989); J. Polchinski, *String Theory* (Cambridge University Press, Cambridge, England, 1998).

[2] B. Zwiebach, Phys. Lett. B **156**, 315 (1985); B. Zumino, Phys. Rep. **137**, 109 (1986).

[3] D. Lovelock, J. Math. Phys. **12**, 498 (1971).

[4] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).

[5] A. H. Chamseddine, Phys. Lett. B **233**, 291 (1989); F. Muller-Hoissen, Nucl. Phys. **B349**,

235 (1990).

[6] C. Lanczos, Ann. Math. **39**, 842 (1938).

[7] D. G. Boulware and S. Deser, Phys. Rev. Lett. **55**, 2656 (1985).

[8] R. G. Cai and K. S. Soh, Phys. Rev. D **59**, 044013 (1999); R. G. Cai, *ibid.* **65**, 084014 (2002).

[9] R. Aros, R. Troncoso, and J. Zanelli, Phys. Rev. D **63**, 084015 (2001).

[10] Y. M. Cho and I. P. Neupane, Phys. Rev. D **66**, 024044 (2002).

[11] J. E. Lidsey, S. Nojiri, and S. D. Odintsov, J. High Energy Phys. **06**, 026 (2002); M. Cvetic, S. Nojiri, and S. D. Odintsov, Nucl. Phys. **B628**, 295 (2002); S. Nojiri and S. D. Odintsov, Phys. Lett. B **521**, 87 (2001); **523**, 165 (2001).

[12] M. H. Dehghani, Phys. Rev. D **67**, 064017 (2003).

[13] T. Levi-Civita, Atti. Accad. Naz. Lincei, Cl. Sci. Fiz., Mat. Nat., Rend. **28** 3 (1919); L. Marder, Proc. R. Soc. London **A244**, 524 (1958).

[14] A. Vilenkin, Phys. Rev. D **23**, 852 (1981); W. A. Hiscock, *ibid.* **31**, 3288 (1985); D. Harari and P. Sikivie, *ibid.* **37**, 3438 (1988); J. R. Gott, Astrophys. J. **288**, 422 (1985); A. D. Cohen and D. B. Kaplan, Phys. Lett. B **215**, 65 (1988); R. Gregory, *ibid.* **215**, 663 (1988); A. Banerjee, N. Banerjee, and A. A. Sen, Phys. Rev. D **53**, 5508 (1996).

[15] B. C. Mukherji, Bull. Calcutta. Math. Soc. **30**, 95 (1938); W. B. Bonnor, Proc. R. Soc. London **A67**, 225 (1954); J. L. Safko, Phys. Rev. D **16**, 1678, (1977); E. Witten, Nucl. Phys. B**249**, 557 (1985).

[16] O. J. C. Dias and J. P. S. Lemos, Class. Quantum Grav. **19**, 2265 (2002).

[17] M. H. Dehghani, Phys. Rev. D **69**, 044024 (2004).

[18] O. Lunin and S. D. Mathur, Nucl. Phys. **B623**, 342 (2002); O. Lunin, J. Maldacena, and L. Maoz, hep-th/0212210

[19] J. H. Horne and G. T. Horowitz, Nucl. Phys. **B368**, 444 (1992).

[20] J. Stachel, Phys. Rev. D **26**, 1281 (1982).

[21] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); E. Witten, *ibid.* **2**, 253 (1998); O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000).

[22] M. Hennigson and K. Skenderis, J. High Energy Phys. **7**, 023 (1998).

[23] V. Balasubramanian and P. Kraus, Commun. Math. Phys. **208,** 413 (1999).

[24] S. Nojiri and S. D. Odintsov, Phys. Lett. B **444**, 92 (1998); S. Nojiri, S. D. Odintsov, and S. Ogushi, Phys. Rev. D **62**, 124002 (2000).

[25] M. H. Dehghani, Phys. Rev. D **66**, 044006 (2002).

[26] M. H. Dehghani, Phys. Rev. D **65**, 124002 (2002); R. G. Cai, *ibid.* **63**, 124018 (2001); A. M. Ghezelbash D. Ida, R. B. Mann and T. Shiromizu, Phys. Lett. B **535**, 315 (2002).

[27] M. H. Dehghani and R. B. Mann, Phys. Rev. D **64**, 044003 (2001); M. H. Dehghani, *ibid.* **65**, 104030 (2002); M. H. Dehghani and H. KhajehAzad, Can. J. Phys. **81**, 1363 (2001).

[28] V. Balasubramanian, P. Horova, and D. Minic, J. High Energy Phys. **05**, 043 (2001); M. H. Dehghani, Phys. Rev. D **65**, 104003 (2001); D. Astefanesei, R. B. Mann, and E. Radu, J. High Energy Phys. **01**, 029 (2004).

[29] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977).

[30] S. Nojiri and S. D. Odintsov, Phys. Rev. D **62**, 064018 (2000); S. Nojiri, S. D. Odintsov, and S. Ogushi, *ibid.* **65**, 023521 (2002).

[31] M. Fukuma, S. Matsuuray, and T. Sakaiz, Prog. Theor. Phys. **105**, 1017 (2001).

[32] R. C. Myers, Phys. Rev. D **36**, 392 (1987).

[33] S. C. Davis, Phys. Rev. D **67**, 024030 (2003).

[34] J. D. Brown and J. W. York, Phys. Rev. D **47**, 1407 (1993).